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Contributions to growth against the previous year of continuous chain-linked Laspeyres indices

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Abstract

The contribution of a subcomponent to the overall growth rate is a popular measure for analysing the effects of subcomponents on an aggregated index. The contributions to growth (CTG) are based on an additive representation of the aggregates' growth in terms of the components' growth. In the case of continuous chained-linked Laspeyres indices, the underlying formulas for CTG covering more than one period have not been explicitly developed yet. The biggest hurdle to this is the additive representation of their growth rates, which are chained expressions themselves and contain several base years and weighting structures. However, the hurdle of one change in the base year and weighting scheme was already cleared by Ribe (1999), who calculates the contribution against the previous year for annually chain-linked Laspeyres indices. In order to develop a formula for CTG against the previous year for continuous chained indices, the steps of Ribes derivation are presented and then modified for continuous chaining. The resulting formula is a special case of CTG according to Ribe and can be easily adapted to calculate the CTG against any previous period.

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1 Purpose and content

This paper develops a formula for calculating year-on-year CTG for continuous chain-linked indices such as the IIE (see Arz and Rentzsch 2018). In order to do so, the basic ideas of the year-on-year growth contribution according to Ribe (1999), which can be used for annually chain-linked Laspeyres indices, are presented in brief and then modified for continuous chaining.

Chapter 2 provides the necessary notation for annually and continuous chain-linked Laspeyres indices. Chapter 3 shows how Ribe derives the year-on-year CTG for annually chain-linked indices. Then, the same steps are applied to derive the year-on-year CTG for continuous chain-linked indices. In chapter 4 an explanation is given why CTG of continuous chain-linked indices are special cases of Ribe's CTG for annually chain-linked indices.

2 Notation

2.1 Annual chain-linking (one-period overlap or annual overlap)

An annual chain-linked index is given by

$$KI^{*A}_{Y,P} = \prod_{y=1}^{Y-1} I^{*A}_{f(y)}\ I^{*A}_{Y,P} \quad (1).$$

$I^{*A}_{f(y)}$ are the annual chain links and $I^{*A}_{Y,P}$ is a chain link for the period P in the current year Y. Each factor in (1) shall be a Laspeyres index. In a period P of year Y, $I^{*A}_{Y,P}$ is given by

$$I^{*A}_{Y,P} = \frac{\sum_a i^a_{Y,P} w^a_{f(Y-1)}}{\sum_a i^a_{f(Y-1)} w^a_{f(Y-1)}} \quad (2).$$

i^a refers to the interesting variable i in the specification a. For example, in the case of a price index, i^a stands for a price of a specific good or service a. w^a is the corresponding weight. $f(Y-1)$ designates the base period, which stems from the respective previous year and is kept constant for all observations within a year.[2]

The chain links $I^{*A}_{f(y)}$ are also constructed according to (2), but with the interesting variables in the nominator stemming from the base period of year Y.

To develop an additive decomposition of growth rates in Chapter 3, it is important that a Laspeyres index can be presented as the weighted sum of its components via:

$$I^{*A}_{Y,P} = \sum_K I^{*K}_{Y,P}\, g^{*K}_{f(Y-1)} \quad (3).$$

[2] For one-period overlap indices (such as the HICPs), the base period $f(Y-1)$ corresponds to the last period Z of the preceding year, which means $f(Y-1) \coloneqq Y-1, Z$. For annual overlap indices (such as the indices of the German national accounts), the prior-year periods average serves as base, so that $f(Y-1) := Y-1$.

The components, or subindices $I_{Y,P}^{*K}$, are also created according to (2), but only for a subset of A. $g_{f(Y-1)}^{*K}$ are the weights of the subindices[3] and sum up to 1.

The subindices may be chain-linked analogously to the aggregate according to (1).

2.2 Continuous chain-linked indices

Continuous chain-linked indices, such as the IEE presented by Arz and Rentzsch (2018), the multi-period index results as

$$KI_{Y,P}^{A} = \prod_{y,p=1,1}^{Y,P} I_{y,p}^{A} \qquad (4).$$

Each chain link is a Laspeyres index, which uses the previous period as the base period:

$$I_{Y,P}^{A} = \frac{\sum_a i_{Y,P}^{a} w_{Y,P-1}^{a}}{\sum_a i_{Y,P-1}^{a} w_{Y,P-1}^{a}} \qquad (5).$$

All periods P smaller than or equal to zero date back to the year before Y. "$Y, 0$" thus corresponds to the last period of the year $Y-1$, "$Y, -1$" is the second last period and so forth.

The index in (5) can also be constructed as a weighted sum of subindices $I_{Y,P}^{K}$ by

$$I_{Y,P}^{A} = \sum_K I_{Y,P}^{K} \, g_{Y,P-1}^{K} \qquad (6).$$

The components $I_{Y,P}^{K}$, are created according to (5). $g_{Y,P-1}^{K}$are the subindex' weights[4] that sum up to 1.

The exchange of K against A in formula (4) provides the continuous chain-linked index of a component.

3 Contributions of the components to year-on-year percentage growth

3.1 Contribution to year-on-year growth for annual chain-linked indices

By using the HICP as an example, Ribe presents the calculation of the effects of components on the year-on-year growth of an aggregated one-period overlap chain Laspeyres-type index. His concept works for other annually chain-linked indices as well. To develop a similar concept for continuous chain-linked indices, it is important to understand the steps of Ribe's derivation.

Firstly, the decomposition bases on the percentage change against the previous year of the aggregate. The year-on-year growth of an annually chain-linked index is

[3] The weights are constructed from the base periods information: $g_{f(Y-1)}^{*K} = \sum_k i_{f(Y-1)}^{k} w_{f(Y-1)}^{k} / \sum_a i_{f(Y-1)}^{a} w_{f(Y-1)}^{a}$.

[4] The weights are constructed from the base periods information: $g_{Y,P-1}^{K} = \sum_k i_{Y,P-1}^{k} w_{Y,P-1}^{k} / \sum_a i_{Y,P-1}^{a} w_{Y,P-1}^{a}$

$$\frac{KI^{*A}_{Y,P}-KI^{*A}_{Y-1,P}}{KI^{*A}_{Y-1,P}} = \frac{KI^{*A}_{Y,P}}{KI^{*A}_{Y-1,P}} - 1 = \ I^{*A}_{f(Y-1)}\frac{I^{*A}_{Y,P}}{I^{*A}_{Y-1,P}} - 1 \qquad (7),$$

(see also Ribe 1999, p. 3). Secondly, Ribe transforms this overall percentage change into a sum of separate growth rates of the years involved. This is done by substituting the short-term index $I^{*A}_{Y,P}$ by the corresponding growth rate $I^{*A}_{Y,P} - 1$. To maintain equality, the difference of $\frac{I^{*A}_{f(Y-1)}}{I^{*A}_{Y-1,P}}$ is added. After rearranging, there results

$$I^{*A}_{f(Y-1)}\frac{I^{*A}_{Y,P}}{I^{*A}_{Y-1,P}} - 1 = I^{*A}_{f(Y-1)}\frac{(I^{*A}_{Y,P}-1)}{I^{*A}_{Y-1,P}} + \frac{I^{*A}_{f(Y-1)}}{I^{*A}_{Y-1,P}} - 1 = \frac{I^{*A}_{f(Y-1)}}{I^{*A}_{Y-1,P}}\left(I^{*A}_{Y,P} - 1\right) + \frac{I^{*A}_{f(Y-1)}-I^{*A}_{Y-1,P}}{I^{*A}_{Y-1,P}} \ (8).$$

On the right-hand side of (8), the term green (second) term forms the development in the previous year ("last-year-term")[5]. The brown (first) term on the right-hand side of (8) indicates the development in the current year Y ("this-year-term"). The factor $\left(I^{*A}_{Y,P} - 1\right)$ is the growth of the chain index in the current year compared to the last year's base period. It is influenced by the chaining factor due to the chain-linking of the index. The denominator normalises the growth to the period of comparison (see also Ribe 1999, p. 6 f.).

Thirdly, Ribe decomposes the "last-year-term" and the "this-years-term" into the growth rates of the components.

The "last-years-term" may be decomposed fully. Please note that the "last-year-term" of each component has the same structure as the "last-year-term" of the aggregate with a substitution of A against K. Arithmetic proves that

$$\frac{I^{*A}_{f(Y-1)}-I^{*A}_{Y-1,P}}{I^{*A}_{Y-1,P}} = \sum_K \left(\frac{I^{*K}_{f(Y-1)}-I^{*K}_{Y-1,P}}{I^{*K}_{Y-1,P}}\right) g^{*K}_{Y-1}\frac{I^{*K}_{Y-1,P}}{I^{*A}_{Y-1,P}} \qquad (9).$$

In "this-year-term", Ribe decomposes only one factor, namely the growth rate of the current year compared to the last base year, $I^{*A}_{Y,P} - 1$:

$$I^{*A}_{Y,P} - 1 = \sum_K \left(I^{*K}_{Y,P} - 1\right) g^{*K}_{Y,P} \qquad (10).$$

The most recent chain link and the denominator must be kept in the aggregated form to ensure additivity of the growth rates (see also Ribe 1999, p. 8 ff.).

Putting it together, (11) shows how much a component contributes to the aggregate growth:

$$CTG^{*K}_{Y-1,P \to Y,P} := \left(I^{*K}_{f(Y-1)} - I^{*K}_{Y-1,P}\right)\frac{g^{*K}_{Y-1}}{I^{*A}_{Y-1,P}} + \frac{I^{*A}_{f(Y-1)}}{I^{*A}_{Y-1,P}}\left(I^{*K}_{Y,P} - 1\right) g^{*K}_{Y-1} \qquad (11).$$

3.2 Contribution to year-on-year growth for continuous chain-linked indices

Firstly, the year-on-year percentage change in a continuous chained-linked index is

[5] The "last-year growth" equals the growth rate of the chained index in the base period of the previous year compared to period P of the previous year.

$$\frac{KI^A_{Y,P}}{KI^A_{Y,P-Z}} - 1 = \prod_{z=1}^{Z} I^A_{Y,P-Z+z} - 1 \qquad (12).$$

Here, Z defines the last period of a year and therefore the number of periods per year. Thus, the chained index of a year ago, $KI^A_{Y-1,P}$, can alternatively be labelled as $KI^A_{Y,P-Z}$. The overall change contains a product with Z one-period-growth factors I^A.

Secondly, the product must be transformed into a sum containing one-period growth rates, the "each-periods-terms". To do so, the last factor $I^A_{Y,P}$ is substituted with the corresponding rate, which equals its value less one. The difference from the original value of the right-hand side is the product of all other factors, which is added:

$$\prod_{z=1}^{Z} I^A_{Y,P-Z+z} - 1 = \underbrace{\prod_{z=1}^{Z-1} I^A_{Y,P-Z+z}}_{\text{Added to maintain equality}} + \underbrace{\prod_{z=1}^{Z-1} I^A_{Y,P-Z+z}\left(I^A_{Y,P} - 1\right)}_{\text{Chain with last factor subsituted by its growth rate}} - 1 \qquad (13).$$

Then, the last factor of the added term is replaced by its growth rate, the difference is added again, and so forth. Only the oldest index $I^A_{Y,P-Z+1}$ does not need to be replaced but may be combined with the "-1" at the end of the line to form a percentage change. A sum results where each summand contains a one-period-growth rate.

$$\begin{aligned}\prod_{z=1}^{Z} I^A_{Y,P-Z+z} - 1 = \quad & \left(I^A_{Y,P-Z+1} - 1\right) + \\ & I^A_{Y,P-Z+1}\left(I^A_{Y,P-Z+2} - 1\right) + \\ & I^A_{Y,P-Z+1} I^A_{Y,P-Z+2}\left(I^A_{Y,P-Z+3} - 1\right) + \\ & \ldots + \\ & \prod_{z=1}^{Z-1} I^A_{Y,P-Z+z}\left(I^A_{Y,P} - 1\right) \qquad (14).\end{aligned}$$

Just like Ribes "this-year-term", each summand contains the chain links up to the period for which the growth rate is calculated.

Thirdly, each aggregated rate of change can be expressed as a sum of the components' rates of change:

$$I^A_{Y,P} - 1 = \sum_K \left(I^K_{Y,P} - 1\right) g^K_{Y,P} \qquad (15).$$

Just like Ribe's "this-year-term", the chain links within each summand are not to be decomposed.

Finally, the contribution of one component to the total year-on-year percentage change is

$$\begin{aligned}CTG^K_{Y,P-Z\rightarrow Y,P} := \quad & \left(I^K_{Y,P-Z+1} - 1\right) g^K_{Y,P-Z+1} + \\ & I^A_{Y,P-Z+1}\left(I^K_{Y,P-Z+2} - 1\right) g^K_{Y,P-Z+1} + \\ & I^A_{Y,P-Z+1} I^A_{Y,P-Z+2}\left(I^K_{Y,P-Z+3} - 1\right) g^K_{Y,P-Z+3} +\end{aligned}$$

$$\dots +$$

$$\prod_{z=1}^{Z-1} I^A_{Y,P-Z+z} \left(I^K_{Y,P} - 1\right) g^K_{Y,P} \qquad (16).$$

The sum of the CTG of all components yields the aggregated growth rate in (12). It is worth mentioning that (16) can be easily modified to calculate CTG for any desired period of time by varying Z to any other number.

4 CTG for continuous chain-linked indices as a special case

Equation (16) has the same structure as Ribes formula in (11), but with more summands. Yet, there are two apparent differences: Firstly, each of Ribe's summands is normed to the Laspeyres index of the period of comparison. Secondly, there is a subtrahend other than 1 in the first summand of (11). In equation (16), however, there are no denominators nor subtrahends other than 1. Both differences stem from the fact that each period of a continuous chain-linked index is a base period. As a consequence, the growth factors between two consecutive periods are always based on a value of 1. Therefore, a normalisation of the summands is superfluous. For the same reason, the growth rate of the chain index in each period can be expressed as simple as $I^A_{Y,P} - 1$, leaving only 1s as subtrahends. If, for an annually chain-linked index, Ribe's period of comparison is set to a base period, denominators and subtrahends other than 1 vanish as well. Consider calculating and expanding

$$\frac{KI^{*A}_{Y,P} - KI^{*A}_{f(Y-2)}}{KI^{*A}_{f(Y-2)}} = I^{*A}_{f(Y-1)} I^{*A}_{Y,P} - 1 = I^{*A}_{f(Y-1)}\left(I^{*A}_{Y,P} - 1\right) + I^{*A}_{f(Y-1)} - 1 \qquad (17).$$

Here, denominators and subtrahends on the right-hand side are all 1. In principle, all growth rates of a continuously chained index follow this special structure.

Literature

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